Random template banks and relaxed lattices

or: why it's efficient to be lazy

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Golm-Han Group meeting, 02 Sep 2008

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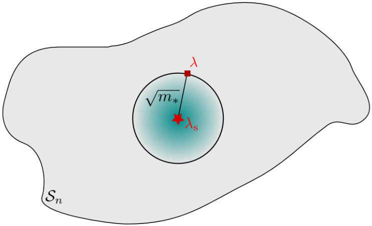
Introduction: metric & template banks

Matched filtering with templates $h(t; \lambda)$



Signal in $\lambda_s \in \mathcal{S}_n$
targeting $\lambda = \lambda_s + \Delta\lambda$

☞ loss: $\mathrm{SNR}(\lambda) < \mathrm{SNR}(\lambda_s)$

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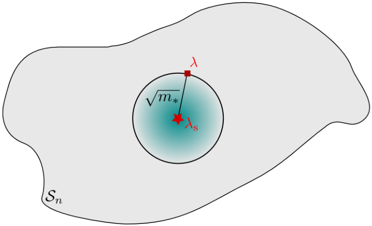

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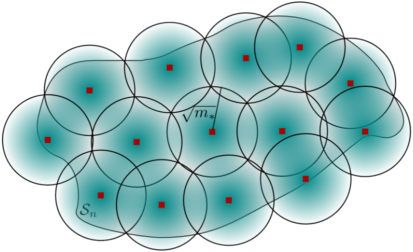

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Each template $\lambda_{(k)}$ *covers* a sphere $|\lambda - \lambda_{(k)}|^2 \leq m_*$

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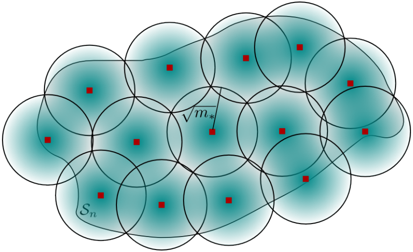

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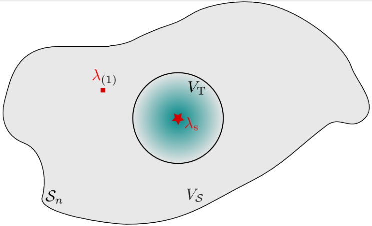
Number of templates: $N = \theta\, m_*^{-n/2}\, V_{\mathcal{S}}$
(normalized thickness θ of a covering)

Random template bank construction



Place one template randomly

$$P(\,\mathrm{miss}\,|\,N_{\mathcal{R}} = 1\,) = 1 - \frac{V_{\mathrm{T}}}{V_{\mathcal{S}}}$$

$$P(\,\mathrm{hit}\,|\,N_{\mathcal{R}} = 1\,) = \frac{V_{\mathrm{T}}}{V_{\mathcal{S}}}$$

(covered volume $V_{\mathrm{T}} = m_*^{n/2} V_n$)

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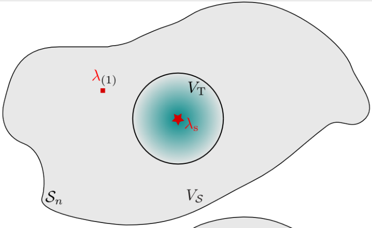

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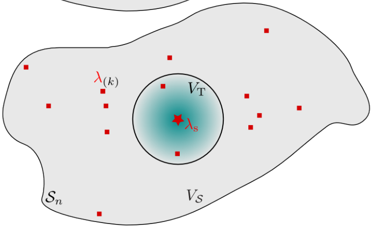

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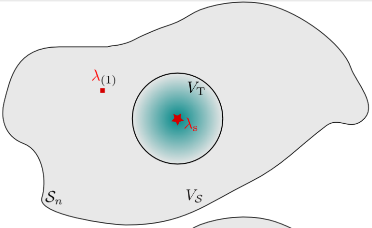

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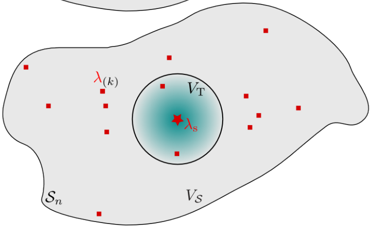

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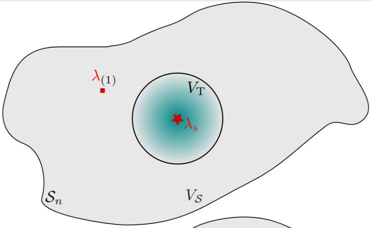

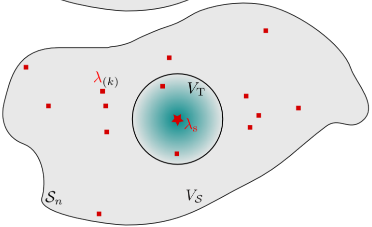

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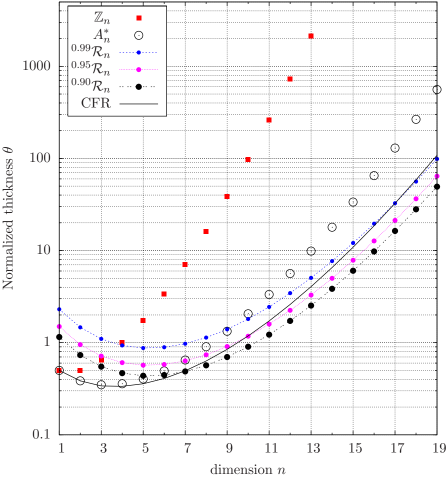
☞ determines $N_{\mathcal{R}}(\eta, m_*) \approx \underbrace{\frac{1}{V_n} \ln\left(\frac{1}{1-\eta}\right)}_{\text{thickness } \theta(\eta)} m_*^{-n/2}\,V_{\mathcal{S}}$

Properties of random template banks

${}^{\eta}\mathcal{R}_n(m_*)$: Random-template bank for mismatch m_* and covering confidence η: thickness $\theta(\eta) = \frac{1}{V_n} \ln\left(\frac{1}{1-\eta}\right)$



- ${}^{\eta}\mathcal{R}_n$ beats best lattice covering at higher dimensions
- beats theoretical lower limit (CFR) on covering thickness
- possible because not strictly a covering
- major practical advantage: simplicity
- if non-constant metric $g_{ij}(\lambda)$: sampling density $\propto \sqrt{\det g_{ij}}$

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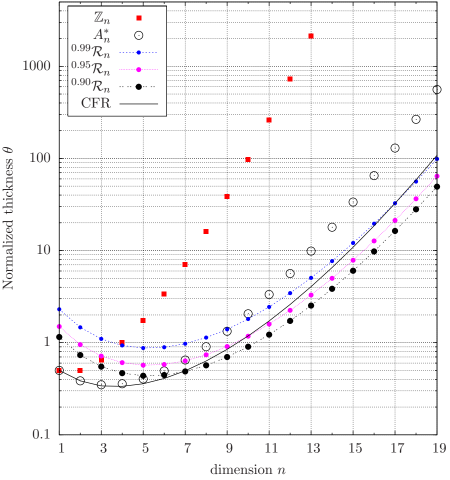

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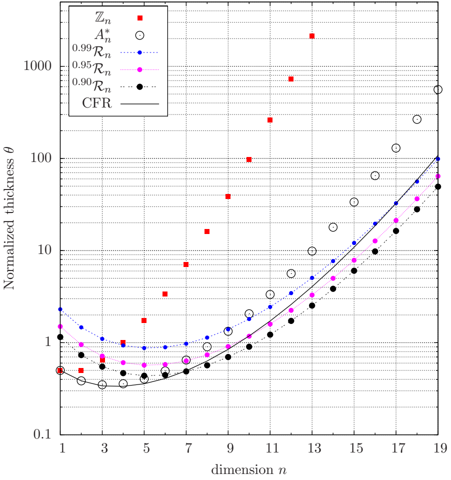

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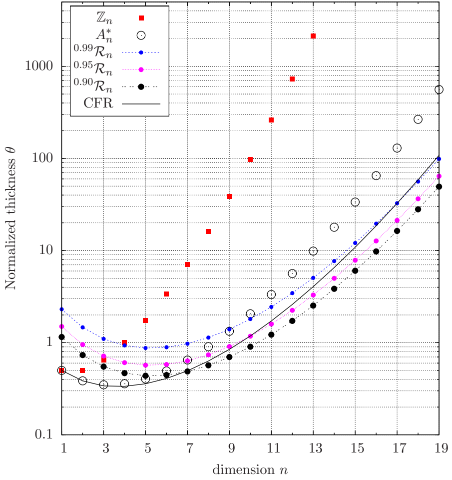

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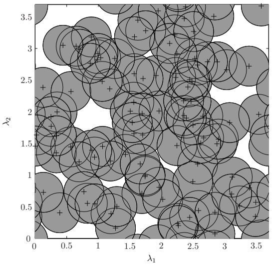
Spatial coverage and worst-case mismatch

example: ${}^{0.9}\mathcal{R}_n(0.1)$



${}^{\eta}\mathcal{R}_n$ provide incomplete coverage

Statistics of individual realizations:

- Spatial fraction $\mathcal{C}$ of parameter-space $\mathcal{S}_n$ covered. Can show $E[\mathcal{C}] = \eta$.
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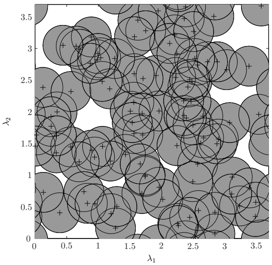

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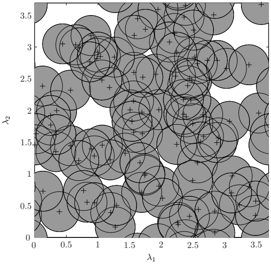

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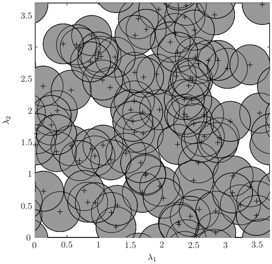

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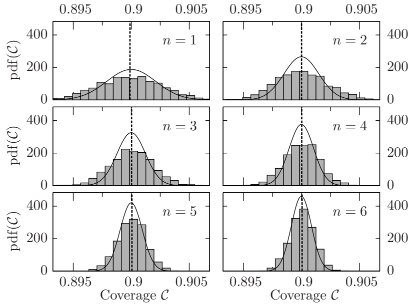

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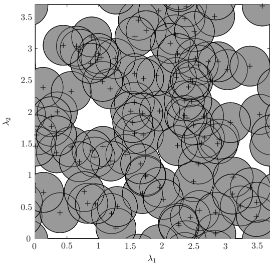

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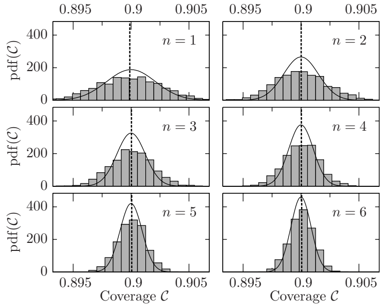

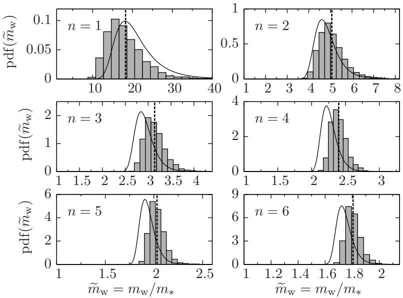

Relaxed lattice coverings

Lesson from ${}^{\eta}\mathcal{R}_n$: efficient to be "lazy". Covering the "last few percent" of $\mathcal{S}_n$ gets increasingly costly at higher dimensions
☞ give up complete coverage for lattices

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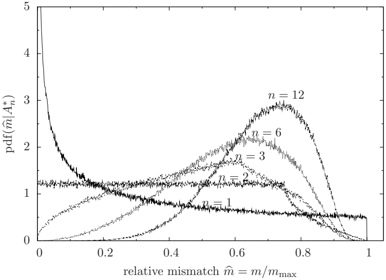

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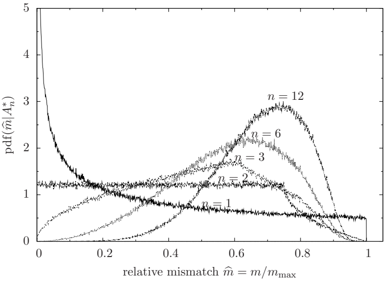

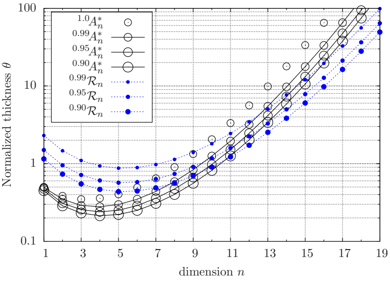

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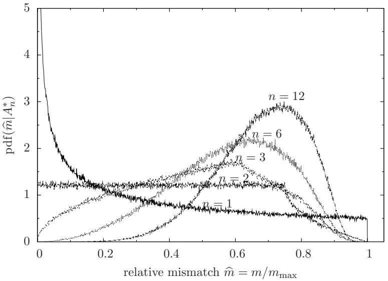

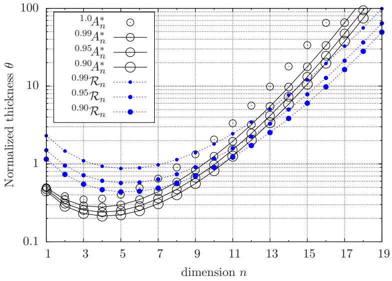

☞ Very efficient, but only at lower dimensions ($n \lesssim 10$)
☞ more difficult than ${}^{\eta}\mathcal{R}_n$, especially with varying metric $g_{ij}(\lambda)$