Estimating the number of templates for a search for short period binary systems

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A quick look at the metric and computation of the number of search templates required for searching for extremely short period binary systems in circular orbits. This is specifically for the analysis of radio data where one may hope to detect a signal from a source in such an system using short data segments.

I. THE METRIC DEFINITION

Let us define our coherent detection statistic, the demodulated power, in the absence of noise as

$$\mathcal{P}(\vec{\Lambda}_0,\vec{\Lambda}) = \frac{A^2}{T}\left|\int_0^T \sin(\phi(t'(\vec{\Lambda}_0)))\exp\left\{-i\phi(t'(\vec{\Lambda}))\right\}\right|^2, \quad (1)$$

$$= \frac{A^2}{4T}\left|\int_0^T \exp\left\{i\Delta\phi(t'(\vec{\Lambda}_0,\vec{\Lambda}))\right\}\right|^2, \quad (2)$$

where $\Delta\phi(t') = \phi(t'(\vec{\Lambda}_0)) - \phi(t'(\vec{\Lambda}))$ represents the phase difference between the true signal described by the true parameters $\vec{\Lambda}_0$ and the template signal described by $\vec{\Lambda}$. Here we are using $t'(\vec{\Lambda})$ to represent the resampled time function. The mismatch $\mu(\vec{\Lambda}_0,\vec{\Lambda})$ we will define as

$$\mu(\vec{\Lambda}_0,\vec{\Lambda}) = 1 - \frac{\mathcal{P}(\vec{\Lambda}_0,\vec{\Lambda})}{\mathcal{P}(\vec{\Lambda}_0,\vec{\Lambda}_0)}, \quad (3)$$

$$= g_{\alpha\beta}(\vec{\Lambda})\Delta\Lambda^{\alpha}\Delta\Lambda^{\beta} + \mathcal{O}\Delta\vec{\Lambda}^3, \quad (4)$$

where we have defined the metric $g_{\alpha\beta}(\vec{\Lambda})$, a quantity that describes the measure of distance within the parameter space. It can be shown that the elements of this metric can be computed via

$$g_{\alpha\delta}(\vec{\Lambda}) = \langle\partial_\alpha\phi\partial_\beta\phi\rangle - \langle\partial_\alpha\phi\rangle\langle\partial_\beta\phi\rangle \quad (5)$$

where $\partial_\alpha\phi$ represents the partial derivitive of ϕ with respect to the α'th parameter and we use the notation $\langle\ldots\rangle$ to represent

$$\langle x(t)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,\mathrm{d}t \quad (6)$$

where t_0 is the start of our coherent observation and T is the duration of this observation.

II. THE PROJECTED METRIC

Projecting the metric on a chosen dimension of the parameter space is advisable if one is to take advantage of possible short-cuts or tricks that enable the search to be performed more efficiently. The standard example is that of the Fast-Fourier-Transform (FFT) where if the frequency dimension can be considered seperate from the others searches can performed on the rest of the space and an FFT performed on each non-frequency template for all frequencies. We can construct the frequency projected metric γ_{ij} using

$$\gamma_{ij} = g_{ij} - \frac{g_{0,i}g_{j,0}}{g_{0,0}}. \quad (7)$$

It can be shown that projection conserves the number of templates required to cover the space (Reinhard), however, it adds some complication to parameter estimation once a signal candidate has been found.

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III. THE NUMBER OF TEMPLATES

The proper volume element of the space is defined as $\sqrt{g(\vec{\lambda})}$ but as we are interested in the number of orbital templates we will use $\sqrt{\gamma(\vec{\lambda})}$. Hence the proper volume of the space is given by

$$\mathcal{V} = \int\int\int \sqrt{\gamma(\vec{\lambda})} \mathrm{d}\Omega \mathrm{d}a \mathrm{d}\psi, \tag{8}$$

and in order to compute the number of templates one has to multiply this by a template density. Template densities are are dependent upon your choice of placement strategy and we will consider the following 3 options 1) cubic lattice 2) A_n^* lattice 3) a random template bank. The template densities are respectively

$$\rho_c = \frac{n^{n/2}}{2^n \mu^{n/2}}, \tag{9}$$

$$\rho_A = \sqrt{\frac{n+1}{\mu^n}} \left[\frac{n(n+2)}{12(n+1)}\right]^{n/2}, \tag{10}$$

$$\rho_r = \log\left(\frac{1}{1-\eta}\right) \frac{\Gamma(\frac{n}{2}+1)}{(\pi\mu)^{n/2}}, \tag{11}$$

where for the random templates *eta* is the fraction of parameter space one wishes to cover with a maximum mismatch μ. The number of orbital templates is then given by

$$N = \rho\mathcal{V}. \tag{12}$$

Note that this is only true if the mismatch ellipses have maximum extent $<<$ the corresponding widths of the parameter space.

IV. THE PHASE MODEL

This bit is simple. We are assuming a highly circularised orbit (not sure on the exact upper-limit on eccentricity) for which the phase can be simplified to

$$\phi(t') = 2\pi f_0 t'(\vec{\Lambda}) + \phi_0, \tag{13}$$

$$t'(\vec{\Lambda}) = t + \tau \sin(\Omega_b t + \psi_0) \tag{14}$$

where ϕ_0 is an initial phase (at $t = 0$), t is measured the observation start time, f_0 is the intrinsic spin frequency of the source (we neglect any higher order spin-down terms), a is the projected orbital semi-major axis normalised by the speed of light, Ω is the orbital angular velocity and ψ represents an initial orbital phase. In this case the search parameter vector is $\vec{\Lambda} = \{f_0, \Omega, a, \psi\}$.

V. THE PARAMETER SPACE BOUNDARIES

In order to define our search space in Ω, a we can ask what are sensible ranges in these parameters given sensible binary masses. As we are looking for neutron stars we can fix one of the binary masses to $1.4M_\odot$. This leaves us with a choice of range for the companion. Given a companion mass m (which in practice can range from $0.1 - 10\,M_\odot$) we can use the following rearrangement of Keplers 3rd law to calculate the regions of parameter space we need to search,

$$a^3\Omega^2 \leq \alpha\frac{Gm^3}{(M_{ns}+m)^2}. \tag{15}$$

Note that we use an inequality here as this is the maximum value of $a^3\Omega^2$ that can be attained which is when the system's orbital plane is parallel to our line of sight. Given that our intention is to fill in the gap that corresponds to orbital periods in the range 5 min - 1 hour this equation then allows us to set the Ω dependent upper bound on the projected orbital semi-major axis. Note that, as is clear from Fig. 1, this boundary and hence the volume of the

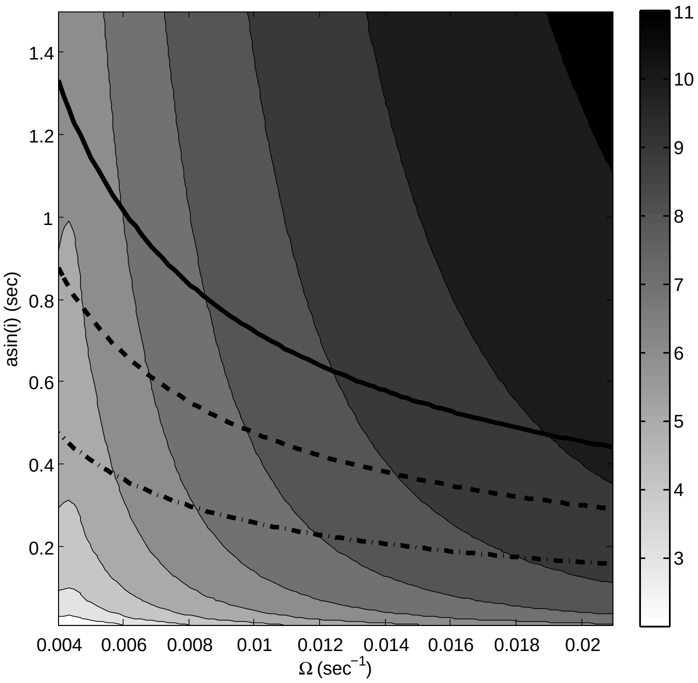

FIG. 1: The Ω, a parameter space plane. The solid line defines the upper boundary for NS's in binary systems with companion mass of $10M_{\odot}$, the dashed line defines the upper boundary for companion masses of $4M_{\odot}$ and the dash-dotted line for comanion masses of $1.4M_{\odot}$. The contours represent the $log_{10}\sqrt{||\gamma||}$ (the proper volume element). A sensible choice of parameter space avoids those regions with high template density. Note that I have limited the minimum range on Ω due to the ill-conditioned nature of the metric at these low values. This can be compensated for by Taylor expanding the metric in this regime and possibly identifying degenerate parameter space dimensions.

search space is strongly dependent upon the maximum companion mass. We can go even further in redicing our space if we place a probablistic bound on the orbital inclination angle. The distribution of the sin of the inclination angle i is uniform on the range $[-1, 1]$ and using this we can further constrain our Ω, a space by searching regions of $a \sin i$ space which contain, say 50% of the systems. In this case we use the parameter α to represent the fraction of systems we wish to be sensitive to at a given maximum companion mass.

	Maximum companion mass ($M_\odot$)		
$P_{\rm orb}$ (min)	1.4	4	10
5	3.5×10^6	2.3×10^7	8.1×10^7
8	2.5×10^5	1.5×10^6	5.3×10^6
10	5.8×10^4	3.8×10^5	1.3×10^6
15	4.1×10^3	2.5×10^4	8.8×10^4

TABLE I: Estimated number of templates required to cover the parameter space for a maximum companion mass and minimum orbital period for an observation time of 268 sec, a mismatch of 0.1, a maximum frequency of 500 Hz and a population fraction $\alpha = 0.5$. This values correspond to using the A_n^* lattice for template placement which at present cannot be applied to a non-flat parameter space such as this one. It should be possible to use a random template bank which would mean an increase in templates by a factor of ~ 1.5.

	Maximum companion NS mass ($M_\odot$)
$P_{\rm orb}$ (min)	1.6
5	5.7×10^6
8	3.5×10^5
10	9.0×10^4
15	6.3×10^3

TABLE II: Estimated number of templates required to cover the parameter space for NS-NS binary system. We have allowed te maximum companion NS mass to be $1.6M_\odot$ and considered the worst case scenario corresponding to the pulsating NS mass lower limit of $1.2M_\odot$. We have maintianed the observation time of 268 sec, a mismatch of 0.1, a maximum frequency of 500 Hz and a population fraction $\alpha = 0.5$. These values again correspond to using the A_n^* lattice for template placement.

VI. THE TEMPLATES

We can now construct the metric as defined in Eq. 5, however each metric element is complicated and (as ever) non-illuminating and is ommitted explicitly here (it would take 3 or 4 pages of tex to show it). What we can do is numerically evaluate the number of templates required (see Table. I) and estimate the scalings associated with the parameter space boundaries.

$$N \propto \left(\frac{f_0}{500\,\mathrm{Hz}}\right)^3 \left(\frac{\mu}{0.1}\right)^{-\frac{3}{2}}. \tag{16}$$

Scalings associated with other choices such as maximum companion mass and minimum orbital period are a bit more complicated so at present see Table. I for a guide. We give in Table. II the template numbers required for searches of the restricted space corresponding to searches for NS-NS binary systems with masses in the range $1.2 - 1.6M_\odot$ and a population fraction 0.5.